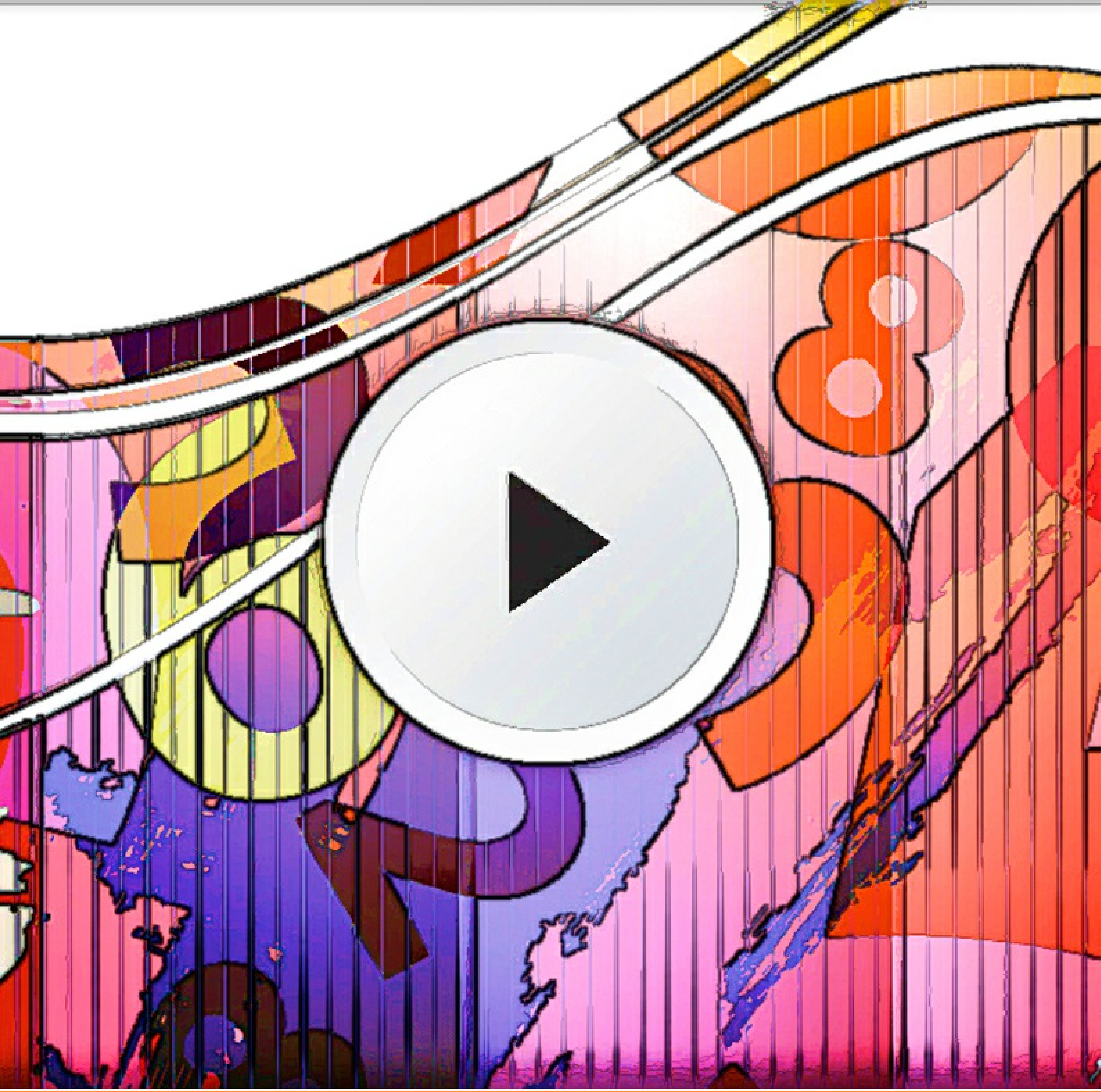

Introduction to Vectors

YouTube classes with Dr Chris Tisdell

Christopher C. Tisdell

bookboon
The eBook company

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Introduction to Vectors

YouTube classes with Dr Chris Tisdell

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Contents

How to use this workbook

This workbook is designed to be used in conjunction with the author's free online video tutorials. Inside this workbook each chapter is divided into learning modules (subsections), each having its own dedicated video tutorial.

View the online video via the hyperlink located at the top of the page of each learning module, with workbook and paper or tablet at the ready. Or click on the *Introduction to Vectors* playlist where all the videos for the workbook are located in chronological order:

Introduction to Vectors
http://www.YouTube.com/playlist?list=PLGCj8f6sgswnm7f0QbRxA6h4P0d1DSD6Q.

While watching each video, fill in the spaces provided after each example in the workbook and annotate to the associated text.

You can also access the above via the author's YouTube channel

Dr Chris Tisdell's YouTube Channel
http://www.YouTube.com/DrChrisTisdell

There has been an explosion in books that connect text with video since the author's pioneering work *Engineering Mathematics: YouTube Workbook* [31]. The current text takes innovation in learning to a new level, with all of the video presentations herein streamed live online, giving the classes a live, dynamic and fun feeling.

About the author

Dr Chris Tisdell is Associate Dean, Faculty of Science at UNSW Australia who has inspired millions of learners through his passion for mathematics and his innovative online approach to maths education. He has created more than 500 free YouTube university-level maths videos since 2008, which have attracted over 4 million downloads. This has made his virtual classroom the top-ranked learning and teaching website across Australian universities on the education hub YouTube EDU.

His free online etextbook, *Engineering Mathematics: YouTube Workbook*, is one of the most popular mathematical books of its kind, with more than 1 million downloads in over 200 countries. A champion of free and flexible education, he is driven by a desire to ensure that anyone, anywhere at any time, has equal access to the mathematical skills that are critical for careers in science, engineering and technology.

At UNSW he pioneered the video-recording of live lectures. He was also the first Australian educator to embed Google Hangouts into his teaching practice in 2012, enabling live and interactive learning from mobile devices.

Chris has collaborated with industry and policy makers, championed maths education in the media and constantly draws on the feedback of his students worldwide to advance his teaching practice.

Acknowledgments

I would like to express my sincere thanks to the Bookboon team for their support.

1 The basics of vectors

1.1 Geometry of vectors

1.1.1 Where are we going?

View this lesson on YouTube [1]

- We will discover new kinds of quantities called "vectors".
- We will learn the basic properties of vectors and investigate some of their mathematical applications.

The need for vectors arise from the limitations of traditional numbers (also called "scalars", ie real numbers or complex numbers).

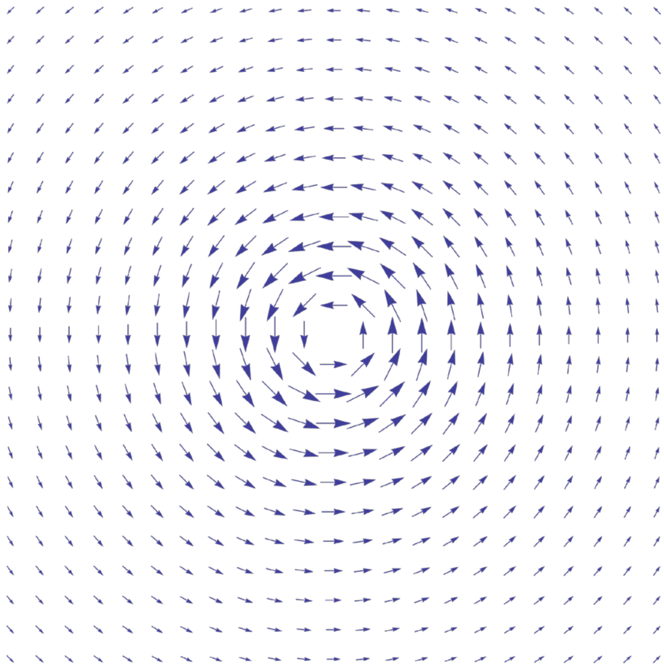
For example:

- to answer the question – "What is the current temperature?" we use a single number (scalar);
- while to answer the question – "What is the current velocity of the wind?" we need more than just a single number. We need magnitude (speed) and direction. This is where vectors come in handy.

1.1.2 Why are vectors AWESOME?

There are at least two reasons why vectors are AWESOME:-

1. their real-world applications;
2. their ability simplify mathematics in two and three dimensions, including geometry.



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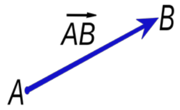
1.1.3 What is a vector?

Important idea (What is a vector?).

A vector is a quantity that has a magnitude (length) and a direction. A vector can be geometrically represented by a directed line segment with a head and a tail.



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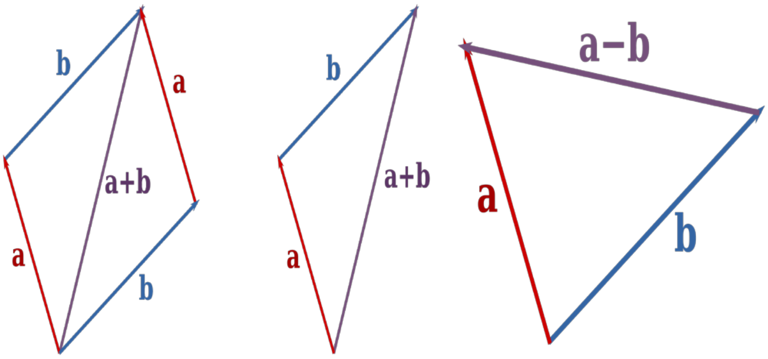
- We can use boldface notation to denote vectors, eg, $\mathbf{a}$, to distinguish the vector $\mathbf{a}$ from the number a.
- Alternatively, we can use a tilde (which is easier to write with a pen or pencil), ie the vector $\underset{\sim}{a}$.
- Alternatively, we can use an arrow (which is easier to write with a pen or pencil), ie the vector $\vec{a}$.
- If we are emphazing the two end points A and B of a vector, then we can write $\vec{AB}$ as the vector from the point A to the point B.

The zero vector has zero length and no direction.

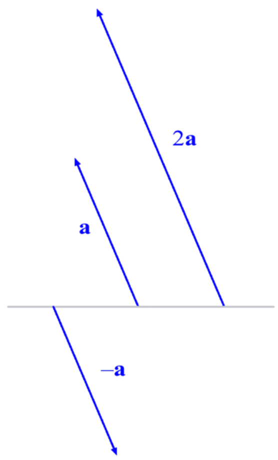
1.1.4 Geometry of vector addition and subtraction



As can be seen from the above diagrams:

- If two vectors form two sides of a parallelogram then the sum of the two vectors is the diagonal of the parallelogram, directed as in the above diagram.
- Equivalently, if two vectors form two sides of a triangle, then the sum of the two vectors is the third side of a triangle.
- Subtraction of two vectors $\mathbf{a}$ and $\mathbf{b}$ involves a triangle / parallelogram rule applied to $\mathbf{a}$ and $-\mathbf{b}$.

1.1.5 Geometry of multiplication of scalars with vectors



As can be seen from the diagram:

- A scalar α times a vector can either stretch, compress and/or flip a vector.
- If $\alpha > 1$ then the original vector is stretched.
- If $0 < \alpha < 1$ then the original vector is compressed.
- If $-1 < \alpha < 0$ then the original vector is flipped and compressed.
- If $\alpha < -1$ then the original vector is flipped and stretched.

1.1.6 Parallel vectors

Important idea (Parallel vectors).

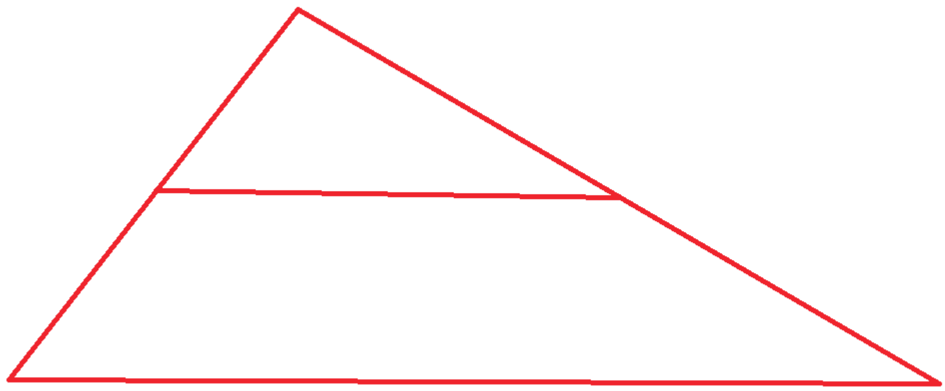
Two non-zero vectors **u** and **v** are parallel if there is a scalar $\lambda \neq 0$ such that

$$\mathbf{u} = \lambda \mathbf{v}.$$

Three points A, B and C will be collinear (lie on the same line) if $\vec{AB}$ is parallel to $\vec{AC}$.

Example.

Consider the following diagram of triangles. Prove that the line segment joining the midpoint of the sides of the larger triangle is half the length of, and parallel to, the base of the larger triangle.



1.2 But, what is a vector?

View this lesson on YouTube [4]

To give a little more definiteness, we can write vectors as columns. Let us take two simple, by very important special vectors as examples:

$$\mathbf{i} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{j} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Any vector (in the xy–plane) can be written in terms of **i** and **j** using the triangle law and scalar multiplication.

Important idea (Column form).

The column form of a vector (in the xy–plane) is

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

For example, $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\mathbf{i} + 2\mathbf{j}$.

1.2.1 How to add, subtract and scalar multiply vectors

> **Important idea** (Basic operations with vectors).
>
> To add / subtract two vectors just add / subtract their corresponding components. To multiply a scalar with a vector, just multiply each component by the scalar.

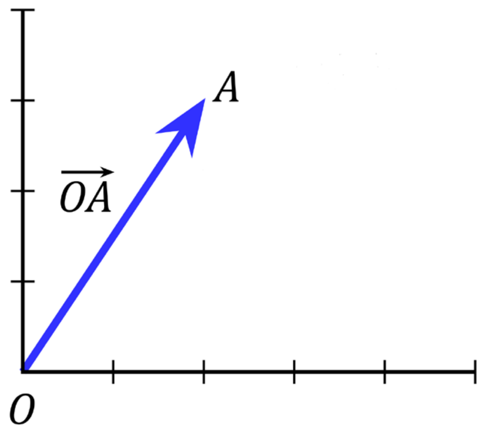
Let $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5\mathbf{i} + 6\mathbf{j}$. Then

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} \\
\mathbf{a} - \mathbf{b} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \\
3\mathbf{a} + 2\mathbf{b} &= 3\begin{pmatrix} 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + \begin{pmatrix} 10 \\ 12 \end{pmatrix} = \begin{pmatrix} 19 \\ 24 \end{pmatrix}
\end{aligned}$$

If we write our vectors in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ then our calculations would look like the following:

$$\begin{aligned} \mathbf{a}+\mathbf{b} &= (3\mathbf{i}+4\mathbf{j})+(5\mathbf{i}+6\mathbf{j}) = 8\mathbf{i}+10\mathbf{j} \\ \mathbf{a}-\mathbf{b} &= (3\mathbf{i}+4\mathbf{j})-(5\mathbf{i}+6\mathbf{j}) = -2\mathbf{i}-2\mathbf{j} \\ 3\mathbf{a}+2\mathbf{b} &= 3(3\mathbf{i}+4\mathbf{j})+2(5\mathbf{i}+6\mathbf{j}) = 19\mathbf{i}+24\mathbf{j}. \end{aligned}$$

If we let the tail point of a vector be at the origin $(0,0)$ and the head point be at the point $A(2,3)$ then the vector formed from this directed line segment is known as the position vector $\mathbf{a}$ of the point A.



Thus, $\mathbf{a} = \vec{OA} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\mathbf{i}+3\mathbf{j}$.

1.3 How big are vectors?

View this lesson on YouTube [5].

To measure how "big" certain vectors are, we introduce a way of measuring the their size, known as length or magnitude.

Important idea (Length / magnitude of a vector).

For a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$ we define the length or magnitude of $\mathbf{a}$ by

$$|\mathbf{a}| := \sqrt{a_1^2 + a_2^2}.$$

Geometrically, $|\mathbf{a}|$ represents the length of the line segment associated with $\mathbf{a}$.

1.3.1 Measuring the direction (angle) of vectors

Using trig and the length of $\mathbf{a}$ we can to compute the angle θ that the vector $\mathbf{a}$ makes with the positive x–axis.

Important idea (Angle to positive x axis).

For a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$, the angle between the vector and the positive x axis is given via

$$a_1 = |\mathbf{a}| \cos\theta; \quad a_2 = |\mathbf{a}| \sin\theta; \quad \tan\theta = a_2/a_1.$$

We take the anticlockwise direction of rotation as the positive direction.

1.3.2 Vectors: length and direction example

Example.

Calculate the length and angle to the positive x axis of the vector

$$\mathbf{a} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \sqrt{3}\mathbf{i} + \mathbf{j}.$$

1.3.3 Properties of the length / magnitude

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1\mathbf{i} + b_2\mathbf{j}$. Some basic properties of the magnitude are:-

$$\begin{aligned} |\mathbf{a}| &= \sqrt{a_1^2 + a_2^2} \geq 0; \\ |\mathbf{a}| &= 0 \quad \text{iff} \quad \mathbf{a} = \mathbf{0}; \\ |\mathbf{a} + \mathbf{b}| &\leq |\mathbf{a}| + |\mathbf{b}|; \\ |\alpha\mathbf{a}| &= |\alpha||\mathbf{a}| \quad \text{where } \alpha \text{ is a scalar;} \end{aligned}$$

The ideas above generalize to more "complicated" situations where the vectors have more components.

1.4 Determine the vector from one point to another point

View this lesson on YouTube [6]

Consider the point $A(1,2)$ and the point $B(4,3)$. What is the vector from A to B? We draw a diagram and apply the triangle rule to see

$$\mathbf{a} + \vec{AB} = \mathbf{b}$$

so a rearrangement gives

$$\begin{aligned} \vec{AB} &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \end{aligned}$$

Important idea (Vector from one point to another).

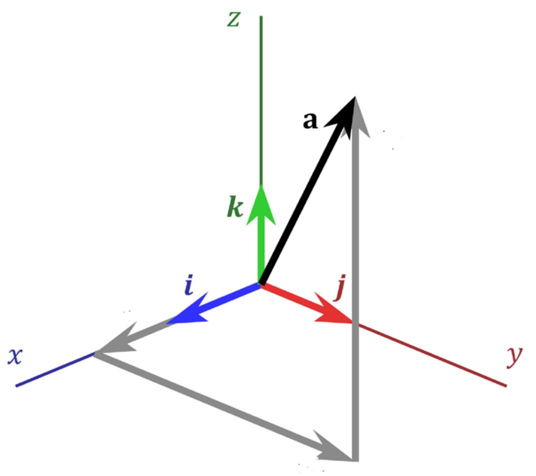
If A and B are points with respective position vectors $\mathbf{a}$ and $\mathbf{b}$ then the vector from A to B is

$$\vec{AB} = \mathbf{b} - \mathbf{a}.$$

The distance between A and B will be $|\vec{AB}|$.

1.5 Vectors in Three Dimensions

View this lesson on YouTube [7]



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Similar to the 2D case, but we now have *three* basis vectors **i**, **j** and a new vector **k** from which we can describe any vector in three-dimensional space.

Important idea (Column form).

The column form of a vector (in $3D$–space) is

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}.$$

For example, $\begin{pmatrix}3\\2\\4\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

Important idea (Length / magnitude of a vector).

For a vector $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ we define the length or magnitude of $\mathbf{a}$ by

$$|\mathbf{a}| := \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

1.5.1 Vectors in higher dimensions

Important idea.

Column form The column form of a vector (in n–dimensional space) is

$$\mathbf{a} = a_1\mathbf{e_1} + \cdots + a_n\mathbf{e_n} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Here the $\mathbf{e_j}$ are unit vectors with all zeros, except for the jth element, which is one. The set of the vectors $\mathbf{e_j}$ are referred to as "the standard basis vectors for $\mathbb{R}^n$".

For a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ we define the length or magnitude of $\mathbf{a}$ by

$$|\mathbf{a}| := \sqrt{a_1^2 + \cdots + a_n^2}.$$

1.6 Parallel vectors and collinear points example

View this lesson on YouTube [2]

Example.

Consider the points: $A(2, -3, 5)$; $B(6, 7, -2)$; $C(-7, 1, 4)$; and $D(-15, -19, 16)$. Calculate the vectors $\vec{AB}$ and $\vec{CD}$. Are they parallel – why / why not? Are A, B and C collinear – why / why not?

1.7 Vectors and collinear points example

View this lesson on YouTube [3]

Example.

Consider the points $A(4, 3, -2)$, $B(-3, -6, 10)$ and $C(25, 30, -39)$.
Compute the vector $\vec{AB}$. Show that the points A, B and C cannot lie on a straight line.

1.8 Determine the point that lies on vector: an example.

View this lesson on YouTube [8]

Example.

Consider the points $A(2,-3,1)$ and $B(8,9,-5)$
Calculate the vector $\vec{AB}$. Determine the point $D(x,y,z)$ that lies between A and B with $\vec{AD} = 2\vec{DB}$.

2 Lines and vectors

2.1 Lines and vectors

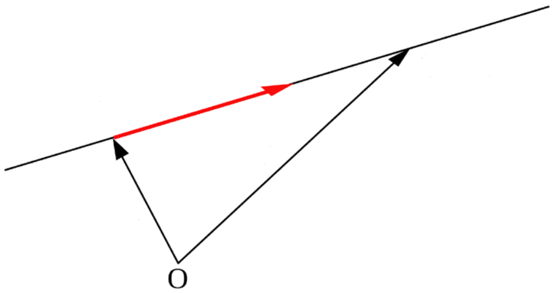
View this lesson on YouTube [9]

We can apply vectors to obtain equations for lines and line segments. For example

Important idea (Parametric vector form of a line).

A line l that is parallel to a vector $\mathbf{v}$ and passes through the point A with position vector $\mathbf{a}$ has equation

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \qquad \lambda \in \mathbb{R}.$$



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2.2 Lines in $\mathbb{R}^3$

View this lesson on YouTube [10]

Let the line l be parallel to $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and pass through the point A with position vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. A parametric vector form for l is

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \qquad \lambda \in \mathbb{R}$$

and we can form an equivalent Cartesian form for the line.

Important idea (Cartesian form of line in $\mathbb{R}^3$).

The Cartesian form for the line l that is parallel the vector $\mathbf{v}$ and passes through the point $A(a_1, a_2, a_3)$ with position vector $\mathbf{a}$ is

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3} \; (= \lambda).$$

2.3 Lines: Cartesian to parametric form

View this lesson on YouTube [11]

Example.

Consider the line l with Cartesian form

$$\frac{x+4}{3} = \frac{y+3}{-2} = \frac{z-3}{-1}.$$

Determine a parametric vector form of the line l. Identify: a point A on l; and a vector $\mathbf{v}$ parallel to l.

2.4 Lines: Parametric and Cartesian forms given two points

View this lesson on YouTube [12]

Example.

Consider the points $A(1, 2, 5)$ and $B(3, 2, 1)$.
Determine a parametric vector form of the line l that passes through A and B. Determine the Cartesian form of l.

2.5 Lines: Convert Parametric to Cartesian

View this lesson on YouTube [13]

Example.

Consider a vector parametric form of a line l given by

$$\mathbf{x} = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Determine the Cartesian form of l. Does the point $(18, 25, 10)$ lie on l and why / why not?

2.6 Cartesian to parametric form of line

View this lesson on YouTube [14]

Example.

Consider the line l with Cartesian form

$$x = 8, \quad \frac{y-3}{2} = \frac{z+5}{-5}.$$

Determine a parametric vector form of the line l. Identify: a point on l; and a vector $\mathbf{v}$ parallel to l.

3 Planes and vectors

3.1 The span of a vector

View this lesson on YouTube [15]

The concept of span is important in connecting the ideas of vectors with lines and planes, plus span arises in many other areas in linear algebra.

The span of a vector $\mathbf{v}$ is connected with all scalar multiples of $\mathbf{v}$, that is

$$\lambda\mathbf{v}, \qquad \text{where } \lambda \in \mathbb{R}.$$

Important idea (Span of a vector).

The equation associated with all scalar multiples of a nonzero vector $\mathbf{v}$ is

$$\mathbf{x} = \lambda\mathbf{v}, \quad \lambda \in \mathbb{R}.$$

The span of $\mathbf{v}$ is the line l that is parallel to $\mathbf{v}$ and passes through the origin.

In set form: $\text{span}(\mathbf{v}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda\mathbf{v}, \text{ for some } \lambda \in \mathbb{R}\}$.

3.1.1 Planes and vectors: span of two vectors

The span of two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ is connected with all "linear combinations" of $\mathbf{v_1}$ and $\mathbf{v_2}$, that is

$$\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2}, \qquad \text{where } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Important idea (Span of two vectors).

The span of two nonzero, nonparallel vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ is the set of points associated with all linear combinations of $\mathbf{v_1}$ and $\mathbf{v_2}$, in set form

$$\text{span}(\mathbf{v_1}, \mathbf{v_2}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2}, \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

The equation

$$\mathbf{x} = \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

describes a plane that is parallel to the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ and passes through the origin.

3.2 Equation of plane: Parametric vector form

View this lesson on YouTube [16]

Combining our ideas on linear combination and span of two vectors, we can now define a plane in $\mathbb{R}^n$.

Important idea (Equation of plane: Parametric vector form).

Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be two nonzero, nonparallel vectors and let A be a point with position vector $\mathbf{a}$. The plane through A that is parallel to $\mathbf{v_1}$ and $\mathbf{v_2}$ has equation

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

This form is known as the parametric vector form of the plane $\mathcal{P}$.

In set form: $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2}, \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}\}$.

The Cartesian form of a (hyper)plane is

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = d$$

where the a_i are constants and d is a constant.

Example.

Consider the Cartesian equation of a plane

$$x - 2y + 7z = 2.$$

Determine a parametric vector form of the plane. Hence, identify two nonzero and nonparallel vectors that are parallel to the plane.

3.3 Planes: Cartesian to parametric form

View this lesson on YouTube [17]

Example.

Consider the Cartesian equation of a plane

$$x + 3y - 2z = 4.$$

Determine a parametric vector form of the plane. Hence, identify two nonzero and nonparallel vectors that are parallel to the plane.

3.4 Equation of plane from 3 points

View this lesson on YouTube [18]

Example.

Consider the points $A(1,2,5)$, $B(3,2,1)$ and $C(-2,1,0)$.
Determine a parametric vector form of the plane $\mathcal{P}$ that passes through A, B and C.
Write down two nonzero and nonparallel vectors that are parallel to $\mathcal{P}$.

4 Dot and cross product

4.1 What is the dot product?

View this lesson on YouTube [19]

We have already seen how to multiply a scalar with a vector. But how can we multiply a vector with a vector and what does it mean? How would it be useful?

Important idea (Dot product).

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. The dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \bullet \mathbf{b} := a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We can connect the dot product with lengths and angles via the Cosine Rule for Triangles to obtain the following.

Important idea (Dot product).

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. The dot product of $\mathbf{a}$ and $\mathbf{b}$ can be written as

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\theta$$

where $\theta \in [0, \pi]$ is the angle between the vectors.

For dimensions higher than three, the above actually enables us to define what we mean by an angle between two vectors via

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}.$$

There are two reasons why the dot product is important: to compute the angle between two vectors; to calculate the "projection" of one vector on another vector.

Example.

Calculate the dot product of the following two vectors

$$\mathbf{a} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

and also calculate the angle θ between the two vectors.

4.2 Orthogonal vectors

View this lesson on YouTube [20]

We can use the dot product to compute the angle θ between two vectors. If $\theta = \pi/2$ then the two vectors are perpendicular to each other. This idea is known as "orthogonality".

Important idea (Orthogonal vectors).

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. If

$$\mathbf{a} \bullet \mathbf{b} = 0$$

then the angle between them is $\theta = \pi/2$ and we say that $\mathbf{a}$ and $\mathbf{b}$ are "orthogonal", "normal" or "perpendicular" to each other.

4.2.1 Orthonormal vectors

If we combine the idea of orthogonality with unit length, then we arrive at the concept of orthonormality.

> **Important idea** (Orthonormal vectors).
>
> Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. If
>
> $$\mathbf{a} \bullet \mathbf{b} = 0, \quad \text{and} \quad |\mathbf{a}| = 1 = |\mathbf{b}|$$
>
> then we call the pair of vectors $\{\mathbf{a}, \mathbf{b}\}$ an "orthonormal set".

This means that the vectors are perpendicular to one another and both have unit length. For example, the set of vectors $\{\mathbf{i}, \mathbf{j}\}$ forms an orthonormal set.

The ideas generalize, for example, the set of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms an orthonormal set, but the sets $\{\mathbf{i}, -\mathbf{i}, \mathbf{k}\}$ and $\{\mathbf{i}, 2\mathbf{j}, \mathbf{k}\}$ do not.

Example.

Determine whether or not the following forms an orthonormal set of vectors.

$$S := \left\{ \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/5 \\ 0 \\ 4/5 \end{pmatrix} \right\}$$

4.3 Scalar Projection of vectors

View this lesson on YouTube [21]

When modelling with vectors, a common question is "What is the force of a given vector in a particular direction?" Two answer this question, we shall first discuss what is known as "scalar projection".

Important idea (Scalar projection).

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. The scalar projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$s_{\mathbf{b}}\mathbf{a} := \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}.$$

Consider the following diagram.

4.3.1 Vector Projection of vectors

If we seek the (vector) force of a given vector in a particular direction, then we have what is known as "the vector projection" of a vector $\mathbf{a}$ on a vector $\mathbf{b}$.

Important idea (Vector projection).

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. The vector projection of $\mathbf{a}$ on $\mathbf{b}$ is

$$\text{proj}_{\mathbf{b}}\mathbf{a} := \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}.$$

Example.

Calculate the scalar and vector projections of

$$\mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \quad \text{on} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}.$$

4.4 Distance between a point and a line in $\mathbb{R}^3$

View this lesson on YouTube [22]

When we speak of distance between a point and line, we mean the MINIMUM distance or perpendicular distance.

Suppose we have a given line l with vector parametric form

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \quad \lambda \in \mathbb{R}$$

and a point B with position vector $\mathbf{b}$.

Important idea (Distance between a point and a line).

The distance from a point B to a line l is

$$|\vec{PB}| = \sqrt{|\vec{AB}|^2 - \left(\frac{\vec{AB} \bullet \mathbf{v}}{|\mathbf{v}|}\right)^2}$$

Example.

Calculate the distance between the point $B(1,2,3)$ and the line

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

4.5 Cross product of two vectors

View this lesson on YouTube [23]

We have seen two kinds of “multiplication” so far with vectors: a scalar multiplying with a vector; and the dot product of two vectors.

Another way of “multiplying” two vectors together is through the idea of a cross product.

Important idea (Cross product).

The cross product of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\mathbf{a} \times \mathbf{b} := (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

The cross product of two vectors produces a new vector that is perpendicular to each of the original vectors.

Sometimes it is easiest to derive this expression using what is known as determinants.

Example.

If

$$\mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

then compute $\mathbf{a} \times \mathbf{b}$.

4.6 Properties of the cross product

View this lesson on YouTube [24]

Here are some properties of the cross product that are sometimes useful in simplifying computations

Important idea (Properties of the cross product).

$$\mathbf{a} \times \mathbf{a} = 0$$
$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \text{so order is important!}$$
$$\mathbf{a} \times \lambda\mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{a} \times \mathbf{b}, \quad \lambda \in \mathbb{R}$$
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}).$$

4.7 What does the cross product measure?

View this lesson on YouTube [25]

We have already seen that the cross product of two vectors produces a new vector that is perpendicular to both of the original vectors.

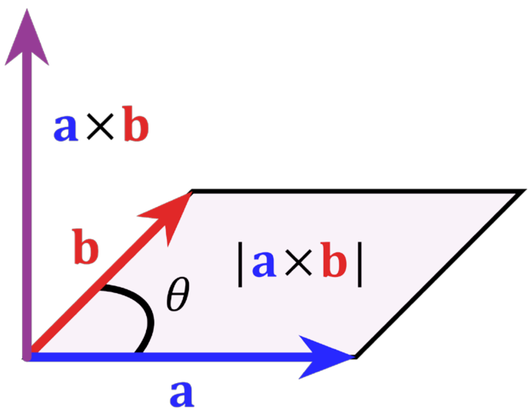
We now relate the cross product with area.

Important idea (Area of parallelogram).

Consider the parallelogram with sides comprised of vectors $\mathbf{a}$ and $\mathbf{b}$. The area of the parallelogram is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\ \sin\theta$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.



4.8 Scalar triple product

View this lesson on YouTube [26]

We can combine two kinds of multiplication of vectors to form the scalar triple product, namely the dot product; and the cross product.

Important idea (Scalar triple product).

The scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}).$$

The computation of the scalar triple product can be performed using the idea of determinants.

Example.

Compute the the scalar triple product $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$, where

$$\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

4.9 What does the scalar triple product measure?

View this lesson on YouTube [27]

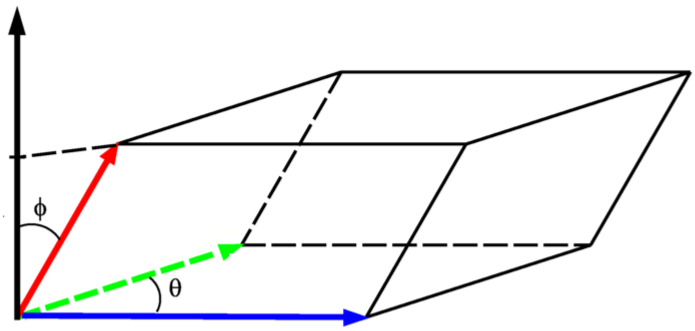
We now relate the scalar triple product with volume.

Important idea (Volume of a parallelpiped).

Consider the parallelpiped with sides comprised of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The volume of the parallelpiped is given by

$$|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|.$$

If $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| = 0$ then the volume is zero and the three vectors must lie in the same plane.



Example.

Compute the volume of the parallelpiped with sides associated with the vectors

$$\mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}.$$

4.10 Equation of plane in $\mathbb{R}^3$

View this lesson on YouTube [28]

Important idea.

Equation of plane: point–normal form Let $\mathbf{n}$ be a vector and let a point A have position vector $\mathbf{a}$. For any point with position vector $\mathbf{x}$ we have the following point–normal form of the equation of the plane which has $\mathbf{n}$ as a normal vector and contains the point A

$$\mathbf{n} \bullet (\mathbf{x} - \mathbf{a}) = 0.$$

This form is known as the point–normal form of the plane $\mathcal{P}$.

Example.

Consider a parametric vector form for a plane $\mathcal{P}$

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Does the point $E(3, 0, -1)$ lie on $\mathcal{P}$? Why / why not?

4.11 Distance between a point and a plane in $\mathbb{R}^3$

View this lesson on YouTube [30]

When we speak of distance between a point and plane, we mean the MINIMUM distance or perpendicular distance.

Suppose we have a given plane with vector parametric form

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

and a point B with position vector $\mathbf{b}$.

Important idea (Distance between a point and a plane).

The distance from a point B to a plane is

$$|\vec{PB}| = \frac{|\vec{AB} \bullet (\mathbf{v_1} \times \mathbf{v_2})|}{|\mathbf{v_1} \times \mathbf{v_2}|}.$$

Example.

Compute the distance between the point $B(1,2,3)$ and the planes with equations:

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R};$$

$$2x - y + z = 5.$$

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